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ABSTRACT

This paper provides some guidance as to what to listen for to help students make sense of expressions in ways that connect to their ideas and honestly address the mathematics of rational numbers. It offers a reasonable initial answer to the question, "Where do students' ideas about fractions and ratios come from, and how can we work productively with them in the classroom?" (KHR)

2

The Development of Students' Knowledge of Fractions and Ratios

John P. Smith III

NO AREA of elementary school mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios, and proportionality. These ideas all express mathematical relationships: fractions and ratios are "relational" numbers. They are the first place in which students encounter numerals like $\frac{3}{4}$ that represent relationships between two discrete or continuous quantities, rather than a single discrete ("three apples") or continuous quantity ("4 inches of rope"). Proportionality concerns an identity between two (or more) such relationships.

Students' experience with these concepts begins early—even before formal schooling—and extends well into the high school years. Before they learn anything in our classrooms, students engage in activities in the everyday world where they generate ideas about fractions, ratios, and proportionality. Some activities take place in the home (e.g., how to divide up an object among friends), some in organized practices in the wider culture (e.g., tracking baseball batting averages). In these activities children construct knowledge about relational numbers. They bring this *constructed* knowledge into our classrooms where it interacts with what our curriculum and teaching offer (*instructed knowledge*). Mathematically successful students manage to connect these two bodies of knowledge; students who never really understand do not. It is our task as teachers to help students connect their constructed knowledge to the powerful new ideas we want to teach them. We must help students make sense of expressions like $\frac{3}{5}$ and $x/3 = 8/12$ in ways that (1) connect to their ideas and (2) address honestly the mathematics of rational numbers. But to do so, we need to listen for their ideas, which are often quite different from what we are teaching and are sometimes

3

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only partially correct. My main purpose in this article is to provide some guidance in what to listen for.

Given the complexity of people's experiences and ideas and of the mathematics of rational numbers, one short article cannot present a complete survey of the nature of students' ideas and how they arise and interact with instruction. The whole story is simply too complex. What this chapter offers is a reasonable initial answer to the question, "Where do students' ideas about fractions and ratios come from, and how can we work productively with them in the classroom?" Because more is known about their development, I shall focus more on the origins of fraction ideas than ratios.

FRACTIONS, RATIOS, PROPORTIONS, AND RATIONAL NUMBERS

To communicate clearly we must be careful about our terminology in this area of mathematics. Mathematics educators often talk right past one another because they are using *fraction*, *ratio*, or *proportion* in different ways without realizing it. My terminology may not strike you as either necessary or optimal, but I want to be clear about my basic terms to avoid miscommunication.

We must first acknowledge that when someone writes a numeral like " $3/4$," we don't know how they are thinking until they tell us. Indeed, *fraction* is such a powerful mathematical idea because it can be used to express so many different kinds of relationships. But the multiplicity of meanings can wreak havoc with our communication. For example, we typically use the same numeral to express both fractions and ratios, yet these ideas are quite different.

To provide some clarity on this point, I will use the term *quotient* to refer to numerals like " $3/4$ " and " $11/7$ " whenever the meaning of the numeral is ambiguous. When the context makes clear that we are referring to a divided quantity, I will use the term *fraction*. *Divided quantity* is a synonym for *partitioned quantity*; both terms refer to a whole quantity that has been divided into some number of equal-sized parts. When a quotient refers to a multiplicative relationship between two quantities (e.g., how many times larger or smaller one is than the other), I will use the term *ratio*. For example, the numeral " $3/4$ " also expresses the relationship between the 12 boys and 16 girls in a classroom of 28 students (i.e., for every 3 boys, there are 4 girls). The word *multiplicative* is essential here because we could also describe the relationship between boys and girls in additive terms, that is, as a difference, "4 more girls."

Similarly, I will use *proportion* and *proportionality* to refer to reasoning with ratios. If ratios are multiplicative relationships between two quantities (as above), then proportional thinking involves projecting the same ratio

into another situation (or situations). Writing equations like " $3/4 = 6/8$ " and " $2/3 = x/12$ " does not necessarily indicate proportional reasoning. In both cases, the person may be thinking about equivalent fractions. As with numerals, equations do not immediately reveal how the people who wrote them are thinking; the writer needs to explain.

All teachers of mathematics have very likely encountered the term *rational numbers*, in their education and teaching. How exactly do rational numbers relate to quotients, fractions, and ratios? In a word, the answer is "Not in any simple way." From the standard mathematical perspective, rational numbers are quotients; they take the form " a/b , where a and b are natural numbers, and $b \neq 0$." One way to think about rational numbers is that they are all the possible solutions to equations of the form, " $ax = b$," where the conditions above apply. Rational numbers have many important properties as a number system. They are dense; they make up an ordered field; they are closed with respect to both addition and multiplication; you can "build" the real numbers with them, to name just a few.

But how do rational numbers relate to fractions and ratios? Here there are important puzzles. For example, we can add ratios but we don't add them the way we add fractions. If " $2/3$ " and " $2/3$ " are ratios, then " $2/3 + 2/3 = 4/6 = 2/3$." And this makes perfect sense—as ratios. If Samantha got 2 hits out of 3 at-bats at both Tuesday's and Friday's baseball games, then her batting average for those two games is $(2 + 2) / (3 + 3)$ or " $4/6$." But as fractions, the same sum is " $4/3$ " or " $1 \frac{1}{3}$." Instead of trying to resolve these puzzles, I would rather suggest a way to think about the connection among these three kinds of numbers. Rational numbers are a powerful mathematical abstraction from fractions and ratios, as we use them in our daily lives. Some properties of ratios, like how we add them, just don't fit the formal properties of rational numbers. If this sounds confusing, rest assured that it is not a simple matter. But in your teaching, a simple principle follows: Work with your students to understand fractions and ratios in their own terms first. If they don't grasp those ideas, the formal concept of rational numbers will make little sense later on.

THE ORIGINS OF FRACTIONS: PARTITIONING AND DIVIDED QUANTITY

As parents and caregivers of young children know, the need for fractions and the development of action sequences to generate them arise quite early in children's social activities with physical objects. Often objects (like cookies) are desirable and scarce and therefore must be divided up and shared. "Fair" sharing leads quickly to the necessity of parts of equal size. Piaget and

his colleagues were among the first to explore the development of children's ability to partition simple objects (circular, square, or rectangular)—that is, how they make equal-sized parts for any given number (Piaget, Inhelder, and Szeminska 1960, especially chapter 12).

Piaget's task involved sharing a make-believe "cake" equally among some number of dolls (each representing another person in the situation). To solve this problem, children had to satisfy three conditions simultaneously: (1) Some part of the cake must be allotted to each doll, (2) the parts must be of equal size, and (3) the parts, taken together, must exhaust the whole. Preschool children, ages 4 and 5, sharing among three dolls would typically satisfy two conditions but violate the third in the process. For example, they would create 3 approximately equal parts but not exhaust the whole. Piaget's most important finding from this work was a developmental sequence of solutions. Children first construct the 2-partition (that is, equal shares for two people) and succeed at this quite early. Soon they can extend their solution to the 4-partition. Then after substantial struggle they can partition into 3 and its multiples and finally into 5 parts and other prime numbers. This sequence is sensible (a 4-partition should be easier than a 3-partition, since "halving" is early achievement), and has been confirmed in later research.

An important cognitive issue in children's partitioning is the interplay between constructing the partition (dividing the object itself or making a pencil drawing on paper) and holding an image of the partition in mind. Both are acts of mental construction, but only one is visible! For small numbers, children who cannot construct the partition may not be able to imagine it either. But for larger numbers, elementary-school-age children often have an image of the correct partition, but they struggle to recreate it on paper—for example, they cannot quite get the parts to come out equal.

Because children's drawings and mental images are linked, teachers can support their mutual development by providing opportunities for both drawing and correcting partitions and imagining them (e.g., by watching teachers build and adjust partitions or by working with flexible materials themselves). The use of manipulative materials, like Fractions Tiles (Jenkins and McLean 1972) and fraction strips, is important here; but we should not quickly link children's representations with such prepartitioned materials to their ability to construct and reason about their own work. As Ball (1992) has pointed out more generally, students' work with manipulative materials does not automatically generate mathematical knowledge. Fraction manipulatives, skillfully used by teachers, can support children's developing knowledge of fractions, but only when these materials are seen as representations of many, many examples of divided quantities.

Indeed middle grades teachers (grades 5–9) may be lucky enough to have students who can already partition (mentally and physically) into any given

number of parts. If so, they can work with those students to develop meanings for fractions from divided quantities. But the prudent course is to check and see what partitions students can handle. Many children who cannot partition relatively easily will struggle with work that relates fraction numerals to divided quantities. Such students would profit much more from additional work on partitioning.

THE DEVELOPMENT OF STUDENTS' REASONING ABOUT FRACTIONS

Speaking very generally, children's knowledge of fractions moves through two broad phases of development: (1) making meaning for fractions by linking quotients to divided quantities and (2) exploring the mathematical properties of fractions as numbers. Indeed the "strange" properties of fractions that often make them so troublesome to learn (e.g., their "unnatural" properties of order and equivalence) can also make them engaging for students. If we include arithmetic operations in "properties"—as we will in this article—this chunk of mathematics represents one of the main instructional targets of the middle years (grades 5–9). It is difficult to teach this content for understanding; students "have it" one minute and lose it the next. A deep appreciation of the ideas and forms of reasoning that students bring to learn fractions will not solve all our instructional problems, but it will help to address them.

Starting with Fractions: Using Quotients to Name Divided Quantities

In some sense, helping students see how to use quotients to show and talk about fractions is not complex or difficult. Once students can partition, they can learn relatively quickly to work with, and talk about, collections of parts in those partitions. This learning is mostly a matter of students acquiring the accepted language for expressing what they are already doing. Once they appreciate the need to talk about collections of parts (that is, "three fourths" not just "one of the fourths"), the path to fractions is clear: The quotient, " $3/4$," represents a collection of 3 parts from an object divided into 4 parts. In conceptual development, it matters little whether we require students to learn the official terms *numerator* and *denominator* right away or temporarily accept "the top number" and "the bottom number." What matters is whether students understand that the meaning of the two numerical components is given by their position.

That said, numerous conceptual hurdles stand between students and a solid understanding of fractions as relational numbers. First, students need

to grasp the key idea that fractions name the relationship between the collection of parts and the whole, not the size of the whole or its parts. For example, in figure 2.1 below, which fractional part is greater?

The best answer is "It depends." In total area, the "fourth" is greater. But if we interpret the question as asking for the greater relative amount, the correct answer is the "third." Students who are just learning fractions will need help seeing that they are looking for the relative, not absolute size or

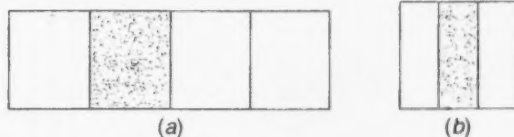


Fig. 2.1. $1/4$ of a larger whole (a) versus $1/3$ of smaller whole (b)

amount. Extensive activity in partitioning wholes in many different ways with many different sized objects is useful here. Care should be given to the language used to describe the relative sizes of parts, collec-

tions of parts, and wholes. For example, references to "big" and "bigger" in relation to divided quantities like those in figures 2.1 and 2.2 can be very confusing for students. All students should be able to explain in clear and convincing terms why the divided quantities in figure 2.2 do or do not represent $2/3$ of the whole.

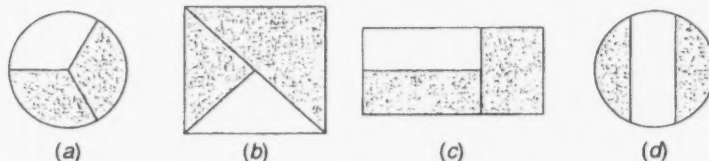


Fig. 2.2. An example (a) and nonexamples (b, c, d) of $2/3$ of the whole

Students also need to become comfortable reasoning and talking about parts of discrete quantities (that is, collections of objects) as fractions. It makes just as much sense to talk about having "three-fourths of a sack of marbles" as it does to talk about "three-fourths of a cookie." Building in work with discrete and continuous quantities helps force the important task of identifying the unit (What is "one whole object" or "one whole set?"). Later on, when students are trying to make sense of arithmetic with fractions, they will need to keep the referent whole in mind.

Third, it is important to wrap language around all work with collections of parts in divided quantities. As students learn to use fractions (words and numbers) to talk relative amounts in divided quantities, they need reminders

and support to be clear about what sort of part it is (e.g., "fifths" or "fourths"), how many there are, and what the referent whole is. As Thompson (1994) has pointed out, drawings do not speak for themselves. The same diagram of a divided quantity can represent many different relationships. In addition to the "standard" interpretation of figure 2.3 below as "3/5 of the whole is shaded," other sensible interpretations include: "1 1/2" (if the whole is the two unshaded parts), "3/2" (for the ratio of shaded to unshaded parts), and "2/3" (for the inverse ratio).

Patient work with students to develop clear and consistent language for divided quantities will resolve some common misconceptions about fractions (e.g., that "2/3" means "two objects, both divided into 3 equal parts" and that "5/4" and other improper fractions are simply "impossible numbers").

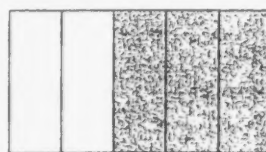


Fig. 2.3. Does this diagram show 3/5? Not necessarily.

Exploring the Mathematical Properties of Fractions

Once students can generate different quantities (continuous and discrete) for given fractions and fractions for different displays of quantity, they are ready to explore fractions as a system of numbers. Two challenging, important, and often fascinating aspects of fractions are their equivalence and order properties. In both cases, fractions defy students' intuitions from natural numbers: There are many different fractions (an infinite number) equal to any given fraction, and between any two fractions there are infinitely many others. Exploring these surprising properties and eventually mastering them will take many years. But the investment in time is worth it for two reasons: (1) Students will understand very little about the arithmetic of fractions if they do not master order and equivalence, and (2) the range of practical application of these ideas in everyday life is vast.

One way into this rich mathematical domain is through the simple question "Which of the following two (or more) fractions is greater, or are they equal?", using carefully selected pairs and groups of fractions. The phrasing of the question is important; it leaves it completely open whether fractions that "look different" really are or just seem to be. I have used this comparison task with much success to study students' understanding of fractions from fifth grade through the college years. I found substantial variation in how students solve the comparison task; there are many different ways of thinking about order and equivalence. Students with the strongest understandings can use many different approaches. To help students connect their

constructed knowledge with what we hope to teach them, we should listen for these different approaches and strategies, support them when we hear and understand them, and help all students build a rich repertoire of fraction strategies.

Consider comparing $\frac{3}{5}$ and $\frac{5}{7}$. Depending on their age, students' first approach to this problem may be to think about the corresponding divided quantities or sketch them on paper. Because both denominators are prime numbers, this is a challenging task, especially if they start with a circular quantity. But upper elementary school students can partition square or rectangular wholes with care and effort. What makes this problem interesting is that imagining or drawing the quantities often does not always resolve the issue; for many $\frac{3}{5}$ and $\frac{5}{7}$ are just "too close together to tell." Some students, not satisfied with visual judgments from their diagrams, will also reason more analytically about the size and number of the parts in each. But with this approach, they may also reach an impasse: "There are three of the fifths and fifths are larger than sevenths, but there are more sevenths!" As with all good problems, resist resolving the tension that can build here. If some students think that $\frac{5}{7}$ is greater because "it looks bigger," ask them to convince others. Someone will eventually see that both fractions (or diagrams) are "missing two parts to equal a whole." Since "fifths are larger than sevenths," $\frac{3}{5}$ must be less than $\frac{5}{7}$ because it is missing more. Also, someone may notice that it is useful to draw a line down the middle of each diagram to see "where one half is." This makes it easier to see that $\frac{3}{5}$ is a bit more than "half" but $\frac{5}{7}$ is still more (in effect, that $\frac{5}{7} - \frac{1}{2} > \frac{3}{5} - \frac{1}{2}$ but in quantitative terms). This move may even help to show that "two and a half fifths" and "three and a half sevenths" are both "half."

From a series of carefully chosen comparison problems (and other tasks), I found that upper elementary, middle, and high school students think about fraction order and equivalence in four basic ways. Many students, especially the younger ones, need to construct and examine divided quantities. But in time they see that they can reason about the size and the number of parts they have constructed, despite the flaws in their diagrams. For example, they may get fed up with trying to draw out "elevenths" and "thirteenths" to decide if $\frac{7}{11}$ or $\frac{7}{13}$ is greater. In their frustration, they come to see that the only thing that matters is that "thirteenths" are smaller than "elevenths," so $\frac{7}{11} > \frac{7}{13}$.

With lots of experience reasoning about fractions with divided quantity diagrams, students can shift almost seamlessly to reasoning directly about the given numerator and denominators. For example, with $\frac{7}{11}$ and $\frac{7}{13}$, some students soon conclude that $\frac{7}{11}$ is greater because "11 is smaller than 13," and when asked, explain that "thirteenths are smaller than elevenths." What is going on here is a shift away from physical parts to the

numbers that represent the parts. Though the concept remains the same, reasoning directly about the numerical component is more efficient. This shift from parts to components allows students to apply their extensive knowledge of whole numbers. When fractions are equivalent (e.g., " $8/16$ " and " $12/24$ "), students can use their knowledge of multiplicative relationships, like "twice" or "half of" and "three times" or "a third of." They will find and talk about these multiplicative relationships both "within" (" $8/16 = 12/24$ " because "8 is half of 16" and "12 is half of 24") and "between" (" $2/3 = 6/9$ " because "2 is one-third of 6" and "3 is one-third of 9"). This reasoning is usually grounded in their thinking about divided quantities. But should you become suspicious that a student is just reciting a rule, ask him or her to show why their reasoning works by drawing diagram.

The third way to think about fractions is where they lie relative to important numerical markers or "reference points." For proper fractions, these reference numbers are "0," " $1/2$," and "1" (though all whole numbers work in similar ways for fractions greater than 1). With a difficult comparison like " $8/11$ " and " $7/15$," some students as early as the upper elementary school years will see that " $8/11$ is greater than $1/2$ and $7/15$ is less than $1/2$," so $8/11$ is greater. This noticing is often supported by the idea that "half" for any denominator can be found by dividing it by 2. If asked to explain, some would go on to say that "half of 11 is 5 and a half" and "half of 15 is 7 and a half," so " $7/15$ is not quite a half." Some students will extend this approach by thinking of $1/4$ and $3/4$ as reference points. The capacity to reason by proximity to numerical reference points also shows up in reasoning with large whole numbers as well. With fractions, it appears to grow out of students' extensive experience with divided quantities (e.g., "9 ninths is a full whole") and their early knowledge of "half" as a multiplicative relationship.

Finally, in the middle school years and beyond, students learn the numerical transformations for fractions, conversion to common denominator, conversion to decimals, and cross-multiplication as quick ways to decide questions of order. In contrast to the constructed strategies sampled above, these strategies are usually learned in school. The problem with these methods is that many students cling to them without understanding why they work. So a good classroom strategy is to make finding a way to explain why common denominator works an on-going problem. Easy "proofs" for conversion to common denominator are accessible to students through divided quantity and multiplication by the right form of 1, and cross-multiplication is simply a disguised part of that strategy. But other interesting conversions are possible: Try having students use a conversion to common numerator and have them explain to themselves why it works!

In presenting this overview of reasoning about fractions through (1) divided quantities, (2) numerical components, (3) reference points, and (4)

numerical conversions, I have focused on the variety of strategies that students learn, both on their own and from our teaching. Students who learn to use many of these strategies are those with the strongest and more durable fraction knowledge—knowledge that will continue to support their growth for many years. But not all students' strategies are flawless.

Many of students' faulty strategies arise from the fundamental mistake of interpreting quotients as pairs of whole numbers (e.g., seeing " $3/5$ " essentially as "3 and 5," in at least some contexts). Faulty "additive equivalence" strategies can appear right alongside "correct" multiplicative ones (e.g., " $3/5 = 5/7$ " because $3 + 2 = 5$ and $5 + 2 = 7$) either within or between. More-complicated misconceptions emerge when our tasks are more demanding. For example, I have asked students if they think there are fractions between $3/5$ and $5/7$. Sarah, who had solved the same comparison ($5/7 > 3/5$) a few minutes earlier, identified many such "in-between" fractions: " $3/6$, $3/7$, ..., $3/12$; $4/2$, $4/3$, $4/4$, ..., $4/12$, and $5/2$, $5/3$, $5/4$, $5/5$, and $5/6$." In trying to deal with differences in both numerator and denominator, Sarah appealed to whole-number order and organized fractions by numerator family, not denominator family. "12" was the largest denominator in her numerator families because rulers were divided in inches! All these faulty strategies have two elements in common. They show students trying to integrate their growing knowledge of fractions with what they know about whole numbers, and they appear right alongside correct strategies. Since many students who do not openly express these strategies are thinking in similar ways, these strategies should be considered—not reprimanded—in the classroom. Pointing students back to divided quantities and letting them apply their idea on accurate diagrams is the best general approach to correction.

BUILDING MEANING FOR FRACTION ADDITION

Even with the decreased emphasis on rational number computation in Standards-based curricula, the total number of hours devoted to teaching (and reteaching) fraction arithmetic in the middle grades is enormous. There are no easy recipes for teaching these difficult procedures, because challenging new ideas like the incommensurability of different units ("thirds" are not the same as "sevenths") underlie them. They take time and patience to master—for both students and teachers. But two general principles have proven useful: (1) vary the kinds of tasks posed to students and (2) build on the foundation of divided quantities—especially with addition.

Conceptually, the meaning of fraction addition follows from the meaning of adding (that is, combining) quantities. With two similar divided quantities, it makes perfect sense to consider their combined size. For positive rational numbers, addition is an increasing operation: Sums are greater than

either addend. This property opens the way to estimation as a precursor to work with the standard procedure. Nearly every middle school teacher is familiar with "add across" error (e. g., " $3/5 + 2/3 = 5/8$ "). Though students who make this mistake may be thinking about fractions as pairs of whole numbers (as above), the error is often justified by appeal to divided quantity: Three parts of 5 and two parts of 3 do, in some sense, combine to yield five parts of 8, but only by ignoring the relative sizes of fifths, thirds, and eighths. This misconception can be undermined directly by asking students to consider the relative sizes of the addends and the sum, but there are also excellent complementary tasks.

Instead of asking for the exact sum, ask instead for an estimate (e. g., "About how much do we have if we add $3/5$ and $2/3$?"). Since both addends are somewhat greater than $1/2$ (reasoning by reference points), students can use their existing knowledge to infer that the sum must be greater than 1 and likely close to $1\ 1/2$. The request for an estimate focuses students' attention more on what is being added (especially the size of the addends) than on the steps in a procedure. Asking for an estimate is also more likely to engage students' knowledge of the real-world quantities and their properties, which would also undermine " $5/8$ " as a possible answer. They must learn to attend to the meaning of addition to correct their inevitable whole-number-based mistakes.

Explicit connections can be built between the addition procedure and the meaning of fractions by examining key steps in the addition procedure. Students naturally question why they have to change the denominators before adding and what makes that "OK." Of the two justifications accessible to middle school students, the first is based on the multiplicative identity: multiplying by a suitable form of "1" leaves the addends unchanged. It works for some students, but its connection to the meaning of fractions is obscure, and it leaves many with a sense that they have been tricked mathematically. The second approach connects the change in the addend denominators to the subdivision of parts in the addend quantities (as illustrated in figure 2.4). Students must still appreciate the need to add commensurate units (i. e., equal-sized parts), but now they can see how those parts are generated.

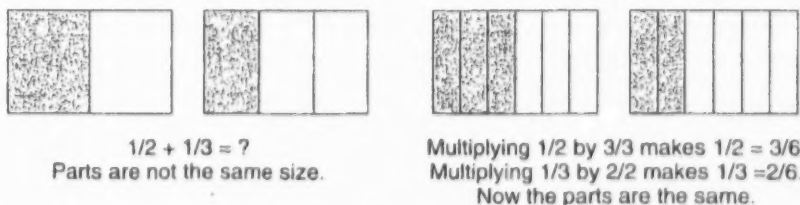


Fig. 2.4. Multiplication (raising terms) as the subdivision of parts

As students begin to work with and understand the addition procedure, consider reversing the direction of their thinking. Instead of giving them the usual series of addition problems (that is, pairs of addends), give them a series of sums and ask them to find the right addends from a set of possibilities. For example, "Find the addends for $2/3$ from the following set, $\{1/2, 1/3, 1/4, \text{ and } 1/6\}$. Instead of thinking algorithmically, they are more likely to think first in terms of the relative size of the addends (e.g., " $2/3$ is greater than $1/2$, so maybe $1/2$ is an addend and I need to look for something small to go with it"). Some students may tend to compute all the possible sums first, especially for small addend sets, but you can make that strategy more difficult by increasing the size of the addend set. This task can easily be extended to subtraction.

THE NATURE OF RATIOS

It is established practice in our mathematical culture to use quotients, numerals of the form a/b , to name ratios as well as fractions. Mathematics educators have struggled to agree on a shared meaning for ratio; indeed, no single consensus definition can be found in the mathematics education literature. I will use the term *ratio* to describe a relational number that has two properties: (1) it relates two quantities in one situation, and (2) it projects that relationship onto a second situation in which the relative amounts of the two quantities remain the same. For example, suppose a soccer team scores 11 goals in 5 games. Someone could think about the ratio "11 goals, 5 games" in order to estimate how many goals the team would score in a 15-game season (that is, 33 goals, 15 games). In the language given above, the first situation is "scoring in 5 games" and the second situation is "scoring by the end of the season." From this perspective, thinking of a ratio is equivalent to what many mathematics educators (and most textbooks) call *proportional reasoning*—with one important exception. People can think of a common ratio extending across the two situations and never write a fraction or an equation between two fractions. Proportional reasoning is a kind of thinking; it is not a matter of what the person writes down on paper.

Ratios are multiplicative, not additive, in nature. In essence, thinking with a ratio in the soccer-season context (that is, thinking of the two situations multiplicatively) involves replicating the 11 goals across each group of 5 games. If there are many such situations in which the quantities stand in the same multiplicative relationship, thinking about a ratio is equivalent to thinking about the linear function $F(x) = mx$ where m is the slope—a very important kind of ratio. In contrast, someone might think of the "goals to games" relationship in the first situation as "6 more (goals than games)."

Applying that to the second situation yields a predicted 21 goals scored (15 games + 6 more). This pattern of thinking (1) links the two situations (like ratio thinking) and (2) is comparative (like ratio thinking), but it is additive in nature, and a wide body of research shows how often middle grades students think additively and incorrectly about situations we see as proportional (e.g., Hart 1988; Lesh, Post, and Behr 1988). These students have conceptualized the same relationship between the two quantities in two different situations, but they have not conceived of that relationship as a ratio.

THE ORIGINS OF RATIOS

The central challenge of developing students' capacity to think with ratios (to reason proportionally) is to teach ideas and restrain the quick path to computation. The research literature on ratio and proportional reasoning shows that students' thinking in situations that we see as proportional often shows they are failing to think about "what is going on" in those situations. In their rush to compute an answer, students do not seem to think long enough about how the quantities in the situation relate and change. Fortunately, there is also evidence that students acquire important foundational ideas that can support more solid understandings of ratios later on.

It appears that some conceptual precursors of ratio appear quite early in children's thinking. Researchers have found that preschool children have intuitive notions of scale (the "large" object in one set belongs with the "large" object in another associated set) and of covariation (if one of two matched objects changes in size, the other must also). These ideas appear to support the later development of elementary school students' sensible additive strategies in proportional situations.

Lamon (1993) and Kaput and West (1994) have studied these additive approaches carefully in middle school students' solutions of "missing value" proportional problems. In both studies, students extensively used "building up" strategies to solve such problems before instruction in proportional reasoning and, to a lesser extent, after. "Building up" the quantities from one situation to equal the quantities in the second is additive thinking, because fundamentally students are adding quantities one group at a time. Although this is not ratio thinking, students are correctly reasoning about the situation—though not yet multiplicatively.

For example, when given the following problem (Kaput and West 1994, p. 245),

A restaurant sets tables by putting 7 pieces of silverware and 4 pieces of china on each placemat. If it used 35 pieces of silverware in its table settings last night, how many pieces of china did it use?

Sixth-grade students typically incremented the set of 4 pieces of china the same number of times as they incremented the set of silverware to reach 35 pieces (that is, 5 times to reach 20 pieces of china). Sometimes the strategy was executed by skip counting, sometimes by saying each ordered pair (e. g., "For 7 silver, there are 4 china; for 14 silver, there are 8 china;" and so on). But at the core, the strategy was a matter of building up larger quantities of silver and china from the combined unit, "7 silver and 4 china" (Lamon 1993). This research has also shown that students' capacity to handle such situations, either by additive or multiplicative reasoning, is strongly influenced by situational factors. Three main factors exert a strong influence on the relative difficulty of such problems: (1) the nature of the situation and students' experience with it (e.g., containment, price, and motion are easier than more abstract situations), (2) the sort of the numbers involved (small whole numbers are easier than larger numbers and rational numbers), and (3) the character of the ratios, within and between corresponding quantities (simple ratios like "2 times" or "3 times as large" are much easier than larger and non-unit ratios). Consequently, it makes little sense to talk about students' capacity to reason with ratios as a unitary ability. Young children, armed with their knowledge of "half," will solve familiar situations with nice numbers quite early; more-difficult situations will stymie students well into high school.

Lamon and Kaput & West also found that their students (mostly sixth and seventh graders) rarely used cross-multiplication to solve missing-value problems, even after the method was taught in class. Although cross-multiplication is both efficient and universally applicable (i.e., it does not depend on specific numbers in the problem), students either do not easily learn it or resist using it when they do. This may be due to the difficulty of linking it to their early knowledge of ratios. The procedure does not match the mental operations involved in the "building up" strategy, and more specifically, the cross-products (e.g., "4 china $\times$ 35 silverware") lack meaning in the situation. What, after all, is one china-silverware, much less 140 of them?

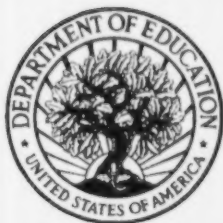
This research leaves us with one important question: How does early additive reasoning about proportional situations support the development of multiplicative reasoning, that is, thinking with ratios? On this crucial point there appears little consensus among researchers and educators, perhaps because the field of proportional situations is so vast. Though researchers are just now addressing this question, two sorts of cognitive capacities seem to support multiplicative interpretations of proportional situations: (1) the ability and willingness to partition and iterate composite units (e.g., if 4 pieces of china are matched to 10 pieces of silverware, seeing that 1 piece of china also matches to 2.5 pieces of silverware), and (2) a focus on relative, not absolute change in situations involving growth and accumulation.

CONCLUSION

Drawing on a diverse body of research, my central message has been that students bring a wide variety of experiences with fractions and ratios and knowledge borne of that experience to their work in our classrooms. Because of the nature of our physical and social world, much of this experience and knowledge centers on dividing (partitioning) and divided quantity. From this foundation grows much of the diversity of students' constructed knowledge and strategies. In our classroom teaching, we ignore this knowledge at our peril. Instead of treating our students as blank slates, we need first to find out how they think and what they know about fractions and ratios, and then begin the process of shaping, reshaping, and building on that knowledge. The cost of ignoring what they bring to our classrooms is that they will learn school procedures for solving school tasks and never integrate their constructed knowledge with the powerful new ways of thinking we have to teach them. If so, they will be poorly prepared to think mathematically in the increasingly complex world of quantities that awaits them.

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